On a numerical algorithm for computing topological characteristics of three-dimensional bodies *

Ya.V. Bazaikin [†]   I.A. Taimanov [‡]

Abstract

We present an algorithm for computing the main topological characteristics of three-dimensional bodies. The algorithm is based on a discretization of Morse theory and uses discrete analogs of smooth functions with only nondegenerate (Morse) and the simplest degenerate critical points.

Keywords: computational topology, cubic complexes, Betti numbers, Morse theory

1   Introduction

The problem of computing topological characteristics of geometrical objects of complicated shape appears in many applications. This led to the rise of computational topology which is fast developing now [1, 2, 3]. Usually a problem of computational topology reduces to effective computation of Betti numbers, i.e. the ranks of homology groups.

One of the most typical approaches to computing homology groups is based on Morse theory [4, 5] which has to be modified for studying objects that are defined by discrete data. Discrete approaches to the Morse theory were systematically studied in [1] and [6] and various discrete realizations of Morse theory were considered in [7, 8, 9, 10].

In this article we propose a method for computing the Betti numbers of three-dimensional bodies. Therewith the bodies are defined as unions of unit

cubes with integer-valued coordinate of vertices in the three-dimensional Euclidean space. The main feature of the approach is

- *the use of discrete analogs of smooth functions (of two variables) which in addition to standard nondegenerate (Morse) critical points have the simplest degenerate critical points of the type of the "monkey saddle" in the case under study.*

The scheme of this approach is as follows. Let M be a three-dimensional body with a boundary in the three-dimensional Euclidean space. We consider the critical points of a differentiable (smooth) function f on M, i.e. the points at which the gradient of f vanishes. The values of f at these points are called *critical*. When the value of f passes through a critical value the topology of the excursion set $\{f \leq \text{const}\}$ changes. By using critical points we construct a chain complex $C_* = C_0 \oplus C_1 \oplus C_2$ which is a graded commutative group. To the critical points there correspond the basic elements (generators) of the complex C_* as of a vector space over the commutative field $\mathbb{Z}_2$ which consists of two elements. To a nondegenerate critical point of index i there corresponds a generator of C_i and to a critical point of the "monkey saddle" type there correspond two generators of C_1. The gradient flow of f defines the boundary operators which are homomorphisms of commutative groups $\partial_i : C_i \rightarrow C_{i-1}$. The final part of this algorithm is standard: since the kernel $\text{Ker}\,\partial$ of a homomorphism $\partial : C_* \rightarrow C_*$ lies in its image $\text{Im}\,\partial$ the homology groups (with coefficients in $\mathbb{Z}_2$) are correctly defined as the quotient spaces

$$H_i = \frac{\text{Ker}\,(\partial_i : C_i \rightarrow C_{i-1})}{\text{Im}\,(\partial_{i+1} : C_{i+1} \rightarrow C_i)}.$$

The dimension of the i-th homology group as of a vector space over $\mathbb{Z}_2$ is called the i-th Betti number of M and it is denoted as b_i. The Betti numbers b_0, b_1, and b_2 which are interpreted as the number of connected components, the number of linearly independent one-dimensional cycles and the number of holes in M and their alternated sum $\chi = b_0 - b_1 + b_2$ are the main topological invariants of M.

Clearly, we consider a discrete version of this algorithm. Note that since a body is three-dimensional it is enough to restrict consideration to homology with $\mathbb{Z}_2$ coefficients for a complete description of homological characteristics of M. Also we may use the Alexander duality and compute only the homomorphism $\partial_1 : C_1 \rightarrow C_0$. We will discuss that here in morein detail.

Since we consider only bounded bodies, every such a body is assumed to be embedded into a cube K in the three-dimensional space. The algorithm

of computing the matrices of boundary operators has complexity $O(n)$ where n is the number of vertices, points with integer-valued coordinates, in K, and the complexity of computing Betti numbers is equal to $O(n_c^3)$, where n_c is the number of critical points of f.

REMARK 1. There exist infinitely many different topological types of critical points of analytic functions of n $(n \geq 2)$ variables and their classification is very complicated. However there are finitely many Morse singularities and topological surgeries of corresponding level surfaces are completely well described. Therefore in computational topology it was supposed to use only the functions with Morse singularities for computing topological characteristics (see the scheme below). However it appears that after discretization the classification of critical points becomes finite which is quite natural and by describing all topological surgeries corresponding to non-Morse singularities one may use more general functions for realization of the scheme mentioned above and we demonstrate that in the three-dimensional case.

REMARK 2. There are algorithms for computing Betti numbers with complexity asymptotically linear in n, [1] which substantially use the specifics of the three-dimensional case (see, for instance, [11]). In applications [12] in which our algorithm was used the number n_c of critical points is small as compared with n and that gives a practical advantage. Moreover the proposed method may be generalized to higher dimensions.

2 Critical points of the diagonal function on a Euclidean cube

By an elementary interval $I \subset \mathbb{R}$ we mean the set $I = [l, l + 1]$ for some $l \in \mathbb{Z}$. We analogously define the elementary square $Q = I_1 \times I_2 \subset \mathbb{R}^2$ and the elementary cube $C = I_1 \times I_2 \times I_3 \subset \mathbb{R}^3$, where I_k are elementary intervals.

We consider three-dimensional bodies M formed by elementary cubes and lying inside the bounded domain

$$K = [0, N] \times [0, N] \times [0, N] \subset \mathbb{R}^3,$$

for some natural number N.

---

[1] The complexity of effective algorithms of distinguishing connected components is equal to $O(n\alpha(n))$ where $\alpha(n)$ is a slowly growing function which is inverse to the the Ackermann function.

In fact we compute homology not of M but of the body $\widetilde{M}$ which is constructed from M by unstacking joint vertices and boundary edges for such elementary cubes, in M, which intersect only at these joint vertices and edges.

We make to remarks:

1) we consider $\widetilde{M}$ due to applications of the algorithm; [2]

2) the body $\widetilde{M}$ itself, or more precisely a topologically equivalent body, may be realized by unions of elementary cubes in a cube K' of a larger size, as shown by the following construction. Let us consider the cube

$$K' = [0, 3N] \times [0, 3N] \times [0, 3N],$$

and the linear map $P(x_1, x_2, x_3) = (3x_1, 3x_2, 3x_3)$ from K to K'. We put $M' = P(M)$, i.e.. M' is a subdivision of M such that every elementary edges splits into three parts and every elementary cube splits into 9 parts. Then we remove from m' all elementary cubes such that they have a joint face with $\partial M'$. We obtain the new cubical set $\widetilde{M}$. In the sequel we assume that M is obtained, if need be by such a surgery.

Let us take for a function f on K the "diagonal" function

$$f(x_1, x_2, x_3) = x_1 + x_2 + x_3.$$

As usual, we put
$$M^a = \{\bar{x} \in M | f(\bar{x}) \leq a\}.$$

It is clear that if $a_1 < a_2$ and the set $f^{-1}([a_1, a_2]) \cap M$ does not contain points with integer-valued coordinates, then M^{a_1} and M^{a_2} are topologically equivalent (see, for instance, [4]). Therefore, under a continuous variation of a from 0 to $3N$ the topological type of M^a changes only when a passes through a value of the form $a = f(x_1, x_2, x_3)$ where $x_i \in \mathbb{Z}$.

Let k_1, k_2, k_3 be a triple of integer numbers such that $v = (k_1, k_2, k_3) \in K$. We put
$$N(v) = \{\bar{x} \in M | |x_i - k_i| \leq 1, i = 1, 2, 3\}.$$

Hence $N(v)$ consists of eight elementary cubes surrounding v and lying in M. For $a = f(v)$ and $|b - a| < \frac{1}{6}$ we put

$$N_v M^b = N(v) \cap M^b.$$

---

[2] In [12] this algorithm is used for evaluation of topological characteristics of oil reservoirs. Such a reservoir is modeled by a three-dimensional body that is a union of three-dimensional cells. Therewith it is assumed that a passing of oil from one cell to another is possible only though the joint two-dimensional face, and there is no passing through a joint vertex or a joint edge. To this end, we have to make a preliminary processing of M.

We say that $v \in M$ is a critical point of a function f if for sufficiently small positive $\varepsilon < \frac{1}{6}$ the sets $N_v M^{a-\varepsilon}$ and $N_v M^{a+\varepsilon}$ are not topologically equivalent.

For distinguishing the critical point type we code the cubes from $N(v)$ as follows. This neighborhood consists of eight elementary cubes and with respect to the central integer-valued point every cube may be defined by changes of the coordinates whose values are either greater or less inside the cube than at v. Hence each cube is coded by a triple of signs "$\pm \pm \pm$". Every such a cube may lie or not lie in M. Hence we may correspond to $N(v)$ the pair of matrices

$$\begin{pmatrix} t_{---} & t_{+--} \\ t_{-+-} & t_{++-} \end{pmatrix} \quad \begin{pmatrix} t_{--+} & t_{+-+} \\ t_{-++} & t_{+++} \end{pmatrix},$$

where $t_{***} = 1$ if the cube lies in M and $t_{***} = 0$ otherwise.

We say that a critical point $v \in M$ is nondegenerate if $N_v M^{a+\varepsilon}$ topologically looks like the body $N_v M^{a-\varepsilon}$ with a handle of index $i, i = 0, 1, 2$, glued to its boundary, i.e. one of the three possibilities holds:

1) (a handle of index 0) $N_v M^{a-\varepsilon}$ is empty and there appears a new connected component $N_v M^{a+\varepsilon}$;

2) (a handle of index 1) neighborhoods of two points on the boundary of $N_v M^{a-\varepsilon}$ are connected by a thickened interval;

3) (a handle of index 2) to a tubular (ribbon) neighborhood of a circle in the boundary of $N_v M^{a-\varepsilon}$ it is glued a thickened two-dimensional disc therewith the boundary of disc is glued to the circle.

We call a critical point *degenerate* otherwise. The number i is called the index of a critical point.

The following theorem is proved straightforwardly by the enumerating combinatorial types of critical points.

Theorem 1 *All possible type of critical points are given in Table 1.*

The types 1–14 correspond to nondegenerate critical points. The last type correspond to the "monkey saddle" which in the smooth case if given by a critical point $(x, y) = (0, 0)$ of the function $f(x, y) = x^3 - xy^2$. The passing through this value results in gluing two handles of index 1.

Let $M_i, i = 0, 1, 2$, be the sets of critical points of index i. Every monkey saddle v contributes to M_1 the two points: the point v itself and its formal "double" v'. For every $i = 0, 1, 2$ let us consider the vector space C_i over $\mathbb{Z}_2$ with generators from M_i. To construct a chain complex we have to define the linear boundary operators $\partial_1 : C_1 \to C_0$ and $\partial_2 : C_2 \to C_1$.

Table 1: Types of critical points

		index
1.	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$	0
2.	$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$	2
3.	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$	1
4.	$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$	1
5.	$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$	1
6.	$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$	1
7.	$\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$	1
8.	$\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$	1
9.	$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$	1
10.	$\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$	1
11.	$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$	1
12.	$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$	1
13.	$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	1
14.	$\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$	1
15.	$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$	the monkey saddle

3 The gradient flow and the boundary operators

The gradient flow enables us to describe a gluing of handles and that allows us to compute the boundary operators in the chain complex C_*. By a trajectory of the gradient flow we mean a sequence of vertices of M which we may pass going down along edges. The difference from the gradient flow of a smooth function consists in an indeterminacy, i.e., from every vertex we may descend into not a unique vertex that lies below. Since we use the Alexander duality, it is enough to define the gradient flow from critical points of index 1 to critical points of index 0.

Let us recall that we compute homology with coefficients in the filed $\mathbb{Z}_2$ which consist of the two elements 0 and 1 meeting the summation rules: $0 + 0 = 1 + 1 = 0, 0 + 1 = 1 + 0 = 1$.

By a tangent vector $\xi = (v, e) \in TM$ we mean a pair consisting of a vertex v and an edge $e \ni v$ which lie in the boundary of M. We say that ξ is increasing if another end of e lies at a higher level of f; otherwise, we say that ξ is a decreasing vector. Moreover on the set of increasing (or decreasing) tangent vectors at a given point we fix a natural order that is induced by ordering of the coordinate directions.

Let v be a critical point of f. If v has index 0, then exactly three tangent vectors are drawn from v and all of them are increasing. If v is a nondegenerate point of index 1, then to every edge of a decreasing tangent vector we assign some label S that is equal to ± 1 and is as follows: The list of all types of critical points shows that all such edges lie in the boundary of M and there are at least two of them. If there are exactly two such edges then we mark one by $+1$ and another by -1. If there are three such edges then two of them belong to an elementary face from the boundary of M and we mark these two edges by the same label, say by $+1$, and then mark the left edge by -1.

MONKEY SADDLE. If v is a monkey saddle, then we add a fictive critical point v', which lies above v with respect to the level surface of f, and also add a fictive edge vv', connecting v and v'. We put that $(v', (1, 0, 0))$, $(v', (0, 1, 0))$, $(v, (0, 0, 1))$ and (v, vv') are increasing vectors and $(v', (0, 0, -1))$, $(v', v'v)$, $(v', (-1, 0, 0))$, and $(v', (0, -1, 0))$ are decreasing vectors. We put that the end of every non-zero vector $(v', (\alpha, \beta, \gamma))$ coincides with the end of $(v, (\alpha, \beta, \gamma))$, $\alpha, \beta, \gamma = -1, 0, 1$.

Let v be a critical point of index 0, and v' be a critical point of index 1. A pair of tangent vectors $\xi = (v, e)$ and $\xi' = (v', e')$ is called a *gradient* pair, if ξ is an increasing vector, ξ' is a decreasing vector, and the following conditions hold: there exists a sequence of increasing tangent vectors (e_i, v_i),

$i = 0, \ldots, k$ such that $e_0 = e$, $v_0 = v$, the edge e_i connects v_i and v_{i+1}, $e_k = e'$, and for every vertex v_i the vector (e_i, v_i) is greater (with respect to the order) than all other tangent vectors which start at this vertex and lie in M.

Exactly in this case we will write that $\xi < \xi'$. We mean by the gradient flow of f on M the family of gradient pairs of tangent vectors:

$$GF(M) = \{(\xi_1, \xi_2) | \xi_i \in TM, i = 1, 2, \xi_1 < \xi_2\}.$$

By definition, a path in M is a family of successively traversed edges from M. The path formed by edges $e_0, \ldots, e_k$ from the definition of a gradient pair is called gradient.

We define the boundary linear operator by the formula:

$$\partial_1 v_1 = v_0^+ + v_0^-, \quad v_1 \in M_1,$$

where

$$v_0^+ \in \{v \in M_0 | \exists \xi = (v, e), \exists \xi_1 = (v_1, e_1), S(e_1) = +1, \xi < \xi_1\},$$

$$v_0^- \in \{v \in M_0 | \exists \xi = (v, e), \exists \xi_1 = (v_1, e_1), S(e_1) = -1, \xi < \xi_1\}.$$

We recall that M_0 and M_1 are the sets of critical points of indices 0 and 1.

Let us remark that this definition is correct because from every critical point of index 1 there are drawn two decreasing tangent vectors labeled by different signs. Therewith we may go down until we achieve (in finitely many steps) a critical point of index 0. Moreover v_0^+ and v_0^- are uniquely defined because we always choose the highest tangent vector. If we consider two gradient paths γ_1 and γ_1' going down to v_0^+ and v_0^-, then together these paths form a path whose neighborhood forms a handle of index 1 attached when the value of f passes through $f(v_1)$ (see §2).

Let us consider the three-dimensional body $M' = K \backslash M$, the complement to M, and a function $h = -f$ on M'. Clearly, $M_i' = M_{2-i}$, $i = 1, 2$; $M_0' = M_2 \cup \{p_0\}$, where p_0 is the corner point of K, where h achieves its minimal value. Let C_i' be the vector space over $\mathbb{Z}_2$ generated by the elements of M_i'. We have the isomorphisms $C_i' = C_{2-i}$, $i = 1, 2$, and $C_0' = C_2 \oplus \mathbb{Z}_2$, where the Z_2 component is generated by p_0. By applying the construction of ∂_1 to h we obtain the boundary operator $\partial_{1\,h} : C_1' \to C_0'$. Let us consider the dual spaces $C_i'^*$, $i = 0, 1, 2$, consisting of $\mathbb{Z}_2$-valued linear functions on C_i'. There are isomorphisms $C_i'^* = C_i'$, $i = 0, 1, 2$, which assign to every Z_2-function on C_i' the formal sum of points from M_i', at which the function does not vanish. The operator $\partial_{1\,h}$ defines the adjoint operator

$$\delta_0 = \partial_{1h}^* : C_0'^* = C_2 \oplus \mathbb{Z}_2 \to C_1'^* = C_1.$$

We put
$$\partial_2 = \delta_0|_{C_2} : C_2 \to C_1.$$

Theorem 2 *The homology groups of the chain complex $C_* = (C_i, \partial_i)$ are isomorphic to the homology groups of M with coefficients in $\mathbb{Z}_2$: $H_i(C_*) = H_i(M; \mathbb{Z}_2)$, $i = 0, 1, 2$.*

Proof. For every critical point of index 1 there exist two paths such that they go down to critical points of index 0 in the directions labeled by different marks. This pair of gradient paths is combined into the path that connects two critical points of index 0 and hence the formal sum of these points belongs to the image of ∂_1. By using the standard reasonings of Morse theory, it is easily proved that if two critical points of index 0 lie in the same connected component of M then they are connected by the chain of such pairs of gradient paths. Since every connected component of M contains at least one critical point of index 0, we have $H_0(M) = C_0/\mathrm{Im}\,\partial_1$. Analogously it is shown that $H^0(M') = \mathrm{Ker}\,\delta_0$. Then we have $\widetilde{H}^0(M') = \mathrm{Ker}\,\partial_2$. Here we recall that the reduced homology groups $\widetilde{H}_i$ meet the equalities $\widetilde{H}_i(M) = H_i(M)$ for $i > 0$ and $H_0(M) = \widetilde{H}_0(M) \oplus \mathbb{Z}_2$; the analogous equalities are valid for the reduced cohomology groups.

Let us recall the Alexander duality theorem (see, for instance, [13]):

Let $A \subset \mathbb{R}^n$ be a compact polyhedron. Then we have the isomorphism:

$$\widetilde{H}_q(\mathbb{R}^n \setminus A; \mathbb{Z}_2) \approx \widetilde{H}^{n-q-1}(A; \mathbb{Z}_2).$$

By definitions and duality, we have $H_2(M) = \widetilde{H}_2(M) = \widetilde{H}^0(M') = \mathrm{Ker}\,(\partial_2)$. Hence the 0-th and the 2-nd homology groups of C_* are isomorphic to the corresponding homology groups of M.

The Euler characteristic $\chi(M)$ of M is equal to the alternated sum of the numbers of cells of any cellular decomposition of a deformation retract of M and, in particular, to $\chi(C_*)$, and it is also equal to the alternated sum of the Betti numbers, i.e. to $b_0 - b_1 + b_2$. Since $\chi(C_*) = b_0(C_*) - b_1(C_*) + b_2(C_*)$, we have

$$\chi(M) = \chi(C_*).$$

It suffices to notice that $b_1(M) = b_0(M) + b_2(M) - \chi(M) = b_0(C_*) + b_2(C_*) - \chi(C_*) = b_1(C_*)$, and, since we consider homology with coefficients in a field, $H_1(M) = H_1(C_*)$. The theorem is proved.

4 An algorithm for computing the boundary operators

The approach described above may be used for constructing an algorithm as follows: Initially we have some array $M = M[i, j, k]$, where $i, j, k = 1, \ldots, N$, whose entries are equal to 0 or 1. Therewith the three-dimensional body M is formed by all elementary cubes $[i, i+1] \times [j, j+1] \times [k, k+1]$ with $M[i, j, k] = 1$.

The preprocessing consists in the following. We stretch the array M in every direction, i.e. replace it by the array M' such that $M[i, j, k] = M'[3i + i_1, 3j + j_1, 3k + k_1]$, $i_1, j_1, k_1 = 0, 1, 2$, $i, j, k = 1, \ldots N$. Then we assign zero values to such elements $M'[i, j, k]$ that correspond to elementary cubes $[i, i+1] \times [j, j+1] \times [k, k+1]$ intersecting the boundary of M' over a two-dimensional face.

At the input of the algorithm we have a preprocessed array M. At the output we obtain the lists $C0$ and $C1$ of critical points of indices 0 and 1 (as it is described above every monkey saddle contributes into $C1$ two elements; these lists give bases for the groups C_0 and C_1) and the matrix $D = D[i, j]$ that defines the boundary operator ∂_1.

To every vertex v we correspond a formally adjoint vertex v'. Let us consider the function Ind which assigns to every vertex $v \in M$ either its index, if the vertex is a critical point of index 0 or 1; either -1, if it is a monkey saddle; either -2 otherwise. The list $C0$ contains the critical points of index 0, and the list $C1$ consists of the critical points of index 1 and the points v and v' which correspond to every monkey saddle v. The function Num returns the number of a critical point of index i in Ci. To every vertex v there corresponds $GF(v)$, the number of a certain critical point of index 0 to which one may go down from v. Finally, to every vertex v the functions $Down\_1$ and $Down\_2$ return the ends of decreasing tangent vectors starting at v, where $Down\_1$ corresponds to the highest vector, or the empty values if there are no such vectors. Therewith to the critical points of index 1 the functions $Down\_1$ and $Down\_2$ return vertices corresponding to decreasing vectors with different signs. If v is a monkey saddle, the we assume that $Down\_1(v)$ and $Down\_2(v)$ are the ends of vectors $(-1, 0, 0)$ and $(0, -1, 0)$ attached to v. For the adjoint vertex we put $Down\_1(v') = v$ and $Down\_2(v')$ is the end of vector $(0, 0, -1)$ attached to v. The description of the algorithm is given in Table 2.

Applying this algorithm twice to M and its complement M', obtain the matrices $D1$ and $D2$ describing the operators ∂_1 and ∂_2. Afterwards the

Table 2: Algorithm

Input: the preprocessed array M
Output: the lists of critical points $C0$ and $C1$, the boundary operator matrix D

```
 1:    D := 0, C0 = ∅, C1 = ∅
 2:    for for all integer-valued vertices v ∈ M do
 3:        if Ind(v) = 0 then
 4:            Add(C0, v)
 5:            GF(v) := {Num(v)}
 6:        if Ind(v) = 1 then
 7:            Add(C1, v)
 8:            GF(v) := GF(Down_1(v))
 9:            if GF(Down_1(v)) ≠ GF(Down_2(v)) then
10:                D(Num(v), Num(GF(Down_1(v)))) := 1
11:                D(Num(v), Num(GF(Down_2(v)))) := 1
12:        if Ind(v) = −1 then
13:            Add(C1, v)
14:            Add(C1, v′)
15:            GF(v) := GF(Down_1(v))
16:            GF(v′) := GF(Down_1(v))
17:            if GF(Down_1(v)) ≠ GF(Down_2(v)) then
18:                D(Num(v), Num(GF(Down_1(v)))) := 1
19:                D(Num(v), Num(GF(Down_2(v)))) := 1
20:            if GF(Down_1(v′)) ≠ GF(Down_2(v′)) then
21:                D(Num(v′), Num(GF(Down_1(v′)))) := 1
22:                D(Num(v′), Num(GF(Down_2(v′)))) := 1
23:        if Ind(v) = −2 then
24:            GF(v) := GF(Down_1(v))
25:    end
```

Betti numbers are found from the formulas:

$$b_0 = \dim\left(C0\right) - \operatorname{rank}\left(D1\right),$$

$$b_1 = \dim\left(C1\right) - \operatorname{rank}\left(D1\right) - \operatorname{rank}\left(D2\right),$$

$$b_2 = \dim\left(C2\right) - \operatorname{rank}\left(D2\right).$$

The complexity of the algorithm for constructing the matrices of the boundary operators, including the preprocessing of data, is equal to $O(n)$ where $n = N^3$ is the number of vertices in K. For a subsequent computation of Betti numbers we have to compute the ranks of Di, for instance, by the Gauss method, and that needs $O(n_c^3)$ operations where n_c is the number of critical points. Generically the relation between n and n_c depends on the initial data.

References

[1] Edelsbrunner, H., Harer, J., and Zomorodian, A.J.: Hierarchical Morse–Smale complexes for piecewise linear 2-manifolds. In: AMC Symposium on Computational Geometry, Discrete & Comput. Geom. **30**:1 (2003), 87–107.

[2] Kaczynski, T., Mischaikow, K., and Mrozek, M.: Computational Homology. Appl. Math. Sci. Series **157**, Springer-Verlag, New York, 2004.

[3] Zomorodian, Afra J.: Topology for Computing. Cambridge University Press, Cambridge, 2005.

[4] Morse, M.: The Calculus of Variations in the Large. American Mathematical Society Colloquium Publication **18**, AMS, Providence, 1934.

[5] Milnor, J.: Morse Theory. Princeton University Press, Princeton, 1963.

[6] Forman, R.: Morse theory for cell complexes. Advances in Math. **134** (1998), 90–145.

[7] Allili, M., Corriveau, D., Derivière, S., Kaczynski, T., and Trahan, A.: Discrete dynamical system framework for construction of connections between critical regions in lattice height data. Journal of Mathematical Imaging and Vision **28**:2 (2007), 99–111.

[8] Corriveau, D., Allili, M., and Ziou, D.: Morse connections graph for shape representation. In: Advanced Concepts for Intelligent Vision Systems, Lecture Notes in Computer Science **3708**, 2005, 219–226.

[9] Gyulassy, A., Bremer, P.-T., Hamann, B., and Pascucci, V.: A practical approach to Morse–Smale complex computation: scalability and generality. IEEE Trans. Vis. Comput. Graph. 2008 Nov-Dec; 14(6):1619–26.

[10] De Floriani, L., Mesmoudi, M.M., and Donovaro, E.: Extraction of critical nets based on a discrete gradient vector field. In: Eurographics 2002. Geometric Algorithms and Visibility, https://diglib.eg.org/EG/DL/Conf/EG2002/short/short86.pdf

[11] Chen, L., and Rong, Y.: Digital topological method for computing genus and the Betti numbers. Topology and its Applications **157** (2010), 1931–1936.

[12] Bazaikin, Ya.V., Baikov, V.A., Taimanov, I.A., and Yakovlev, A.A.: Numerical analysis of topological characteristics of three-dimensional geological models of oil and gas fields. [Russian] Mathematical Modelling (to appear).

[13] Spanier, E.H.: Algebraic Topology. Springer Verlag, New York–Berlin, 1981.